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Approximation in Sobolev Spaces by Kernel Expansions[1]

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For interpolation of smooth functions by smooth kernels having an expansion into eigenfunctions (e.g., on the circle, the sphere, and the torus), good results including error bounds are known, provided that the smoothness of the function is closely related to that of the kernel. The latter fact is usually quantified by the requirement that the function should lie in the "native" Hilbert space of the kernel, but this assumption rules out the treatment of less smooth functions by smooth kernels. For the approximation of functions from "large" Sobolev spaces W by functions generated by smooth kernels, this paper shows that one gets at least the known order for interpolation with a less smooth kernel that has W as its native space.

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1. INTRODUCTION

Let $\{\varphi_j(x)\}_{j \in \mathscr{I}}$ be a complex-valued orthonormal basis of $L_2(\Omega)$, where $\mathscr{I}$ is a countable index set, Ω is a bounded domain in $\mathbb{R}^n$, or a compact n-dimensional Riemannian manifold [5]; the n-sphere $\mathbb{S}^n$ and the n-torus $\mathbb{T}^n$ are manifolds of special interest.

Expansions of functions $f \in L_2(\Omega)$ with respect to $\{\varphi_j(x)\}_{j \in \mathscr{I}}$ will be written as

$$f = \sum_{j \in \mathscr{I}} \hat{f}(j)\, \varphi_j, \qquad \hat{f}(j) := (f, \varphi_j)_2. \tag{1}$$

The symbols c and C will stand for generic constants. We also wish to define Sobolev-type subspaces $\mathscr{S}_w$ of $L_2(\Omega)$. We let

$$\mathscr{S}_w := \left\{ f = \sum_{j \in \mathscr{I}} \hat{f}(j)\, \varphi_j,\ \|f\|_w^2 := \sum_{j \in \mathscr{I}} \frac{|\hat{f}(j)|^2}{w_j} < \infty \right\} \tag{2}$$

for any sequence $w = \{w_j\}_{j \in \mathscr{I}}$ of positive weights. Of course, for the underlying manifolds involved, the Sobolev spaces are defined in the same way, but with special weights depending on an *order* σ, see (20) and (25). However, to avoid possible confusion, we will write W_σ for the usual order σ Sobolev space.

We shall study approximation of functions $f \in L_2(\Omega)$ by linear combinations of functions $\Phi(\cdot, y)$, where $y \in \Omega$ and $\Phi\colon \Omega \times \Omega \to \mathbb{R}$ is a *symmetric positive definite kernel* (see, e.g., [5, 6, 7]) having an expansion

$$\Phi(x, y) := \sum_{j \in \mathscr{I}} \hat{\Phi}(j)\, \varphi_j(x)\, \overline{\varphi_j(y)} \tag{3}$$

with the coefficients $\hat{\Phi}(j)$ being strictly positive. Such a framework may be viewed as the natural analogue in Ω of RBF approximation on all of $\mathbb{R}^d$. The smoothness of the kernel and the summability of the above series is usually controlled by conditions on the decay of $\hat{\Phi}(j)$ of the form

$$c\, \|j\|^{-\tau} \leqslant \hat{\Phi}(j) \leqslant C\, \|j\|^{-\tau} \tag{4}$$

for $\|j\| \to \infty$, where $\|j\|$ will be a norm on the index set. The precise inequalities in (4) will be provided later in specific cases.

We call a kernel of the form (3) *admissible*, if the sequence $\{\hat{\Phi}(j)\}_{j \in \mathscr{I}}$ satisfies

$$\sum_j \hat{\Phi}(j)\, |\varphi_j(x)|^2 \leqslant C < \infty$$

for all $x \in \Omega$. According to the overview given in [7], there are many admissible kernels arising from positive integral operators

$$v \mapsto \int_{\Omega} v(x)\, \Phi(\cdot, x)\, dx \tag{5}$$

having $\{\varphi_j\}_{j \in \mathscr{I}}$ as a complete orthonormal set of eigenfunctions with eigenvalues $\hat{\Phi}(j)$.

Any admissible kernel generates a Hilbert subspace $\mathscr{S}_{\hat{\Phi}}$ of $L_2(\Omega)$, called the *native space* for Φ, by (2) for the special weights $w_j = \hat{\Phi}(j)$. There is a well-developed theory for interpolation of functions f in the native space (see [1, 2, 5] for the torus and the sphere), but for smooth kernels Φ one usually has fast decay of the eigenvalues $\hat{\Phi}(j)$ of the positive integral operator (5) leading to an undesirably small native space in which interpolation is known to work well. To overcome this drawback, this paper treats the approximation of functions f from fixed large Sobolev-type spaces $\mathscr{S}_w$ via sufficiently smooth kernels Φ of the above form. The final goal of this paper is to prove that the approximation quality is the same as it is for interpolation in the space $\mathscr{S}_w$ itself, if we reinterpret this space as a native space $\mathscr{S}_w = \mathscr{S}_{\hat{\Psi}}$ for a much less smooth, but still admissible kernel Ψ. In particular, we want to consider approximations of functions f from Sobolev spaces W_σ by functions Φ with the decay conditions (4) for $\sigma < \tau$, and we want to obtain at least the approximation power that is attainable for interpolation in W_σ.

2. METHOD OF APPROXIMATION

In this section, we describe the approach taken to obtain the desired rates of approximation. Recall that we wish to approximate a function f of the form (1) from a Sobolev-type subspace $\mathscr{S}_w \subset L_2(\Omega)$ as defined in (2). Since we intend to use interpolation by a smooth kernel Φ whose native space $\mathscr{S}_{\hat{\Phi}}$ does *not* contain $\mathscr{S}_w$, we look for an intermediate approximation that lies in the native space $\mathscr{S}_{\hat{\Phi}}$. A natural candidate is a "cut-off" function of the form

$$f_L := \sum_{j \in \mathscr{I}_L} \hat{f}(j)\, \varphi_j, \tag{6}$$

where $\mathscr{I}_L$ is a sequence of finite subsets of $\mathscr{I}$ constructed so that $\mathscr{I}_L \subset \mathscr{I}_{L+1}$ and $\bigcup_L \mathscr{I}_L = \mathscr{I}$. The $\mathscr{I}_L$'s will depend on specifics of the case being treated.

After we choose the $\mathscr{I}_L$'s, our first step will be to derive error bounds of the form

$$\|f - f_L\|_{\infty, \Omega} \leqslant \alpha(L, w)\, \|f\|_w \tag{7}$$

for the truncation error $f - f_L$. When, in particular, the weights $w(j)$ are those of a Sobolev space W_σ, then we expect that asymptotically

$$\alpha(L, w) = \mathcal{O}(L^{n/2-\sigma}), \tag{8}$$

where n is the dimension of Ω.

As a second step, we interpolate f_L at a finite number of scattered points $X := \{x_1, \ldots, x_N\} \in \Omega$ by a function $I_{\Phi, X}(f_L)$. The set X has an associated mesh norm (or "fill distance")

$$h := \max_{y \in \Omega} \min_{x \in X} \operatorname{dist}(x, y),$$

where $\operatorname{dist}(x, y)$ is the distance between x and y relative to the metric on Ω.

There is a connection between h and the cardinality of X for sets in which the points are quasi-uniformly distributed; that is, the smallest nearest-neighbor distance between points in X is comparable to the largest nearest-neighbor distance. Such sets are optimal in the sense that they have the smallest cardinality compatible with a given mesh norm, and it is easy to show that they satisfy $\operatorname{card}(X) = \mathcal{O}(h^{-n})$, where n is the dimension of Ω.

The standard error bounds for this interpolation process have the form

$$\|f_L - I_{\Phi, X}(f_L)\|_{\infty, \Omega} \leqslant \beta(h, \Phi) \, \|f_L\|_\Phi, \qquad 1 \leqslant p \leqslant \infty, \tag{9}$$

where again for purposes of illustration, we assume

$$\beta(h, \Phi) = \mathcal{O}(h^{\tau - n/2}) \qquad \text{for} \quad h \downarrow 0 \tag{10}$$

in case of a kernel Φ satisfying (4), and where n is the dimension of Ω. We shall comment on special instances of these results later.

As a third and final step one needs to estimate $\|f_L\|_\Phi$ appearing in (9). We derive the following "inverse" bound,

$$\begin{aligned} \|f_L\|_\Phi^2 &= \sum_{j \in \mathcal{I}_L} \frac{|\hat{f}(j)|^2}{\hat{\Phi}(j)} \\ &\leqslant \|f\|_w^2 \max_{j \in \mathcal{I}_L} \frac{w_j}{\hat{\Phi}(j)} \\ &=: \|f\|_w^2 \, \gamma^2(L, w, \Phi). \end{aligned} \tag{11}$$

For the examples we have in mind, suitable bounds for γ will be of the form

$$\gamma(L, w, \Phi) = \mathcal{O}(L^{\tau - \sigma}) \qquad \text{for} \quad L \to \infty \tag{12}$$

in case of weights w coming from a Sobolev space W_σ, and if $\hat{\Phi}$ satisfies (4) with $\sigma < \tau$.

Thus, the approximation scheme yields the error estimate

$$\begin{aligned}\|f-I_{\Phi,X}(f_L)\|_{\infty,\Omega} &\leqslant \|f-f_L\|_{\infty,\Omega}+\|f_L-I_{\Phi,X}(f_L)\|_{\infty,\Omega}\\ &\leqslant \alpha(L,f,w)\,\|f\|_w+\beta(h,\Phi)\,\|f_L\|_\Phi\\ &\leqslant (\alpha(L,f,w)+\gamma(L,w,\Phi)\,\beta(h,\Phi))\,\|f\|_w. \end{aligned}\tag{13}$$

Next, one needs to choose L in terms of h so as to let the two inner terms appearing in (13) have the same asymptotics for $h \downarrow 0$, namely

$$\alpha(L,f,w)\approx\gamma(L,w,\Phi)\,\beta(h,\Phi).\tag{14}$$

For the case where w comes from a space W_σ, and if $\hat{\Phi}$ satisfies (4), then by (8), (10), and (12) we see that the quantity $\|f-I_{\Phi,X}(f_L)\|_{\infty,\Omega}$ behaves like

$$L^{n/2-\sigma}+L^{\tau-\sigma}h^{\tau-n/2}$$

and so it is clear that L must grow inversely proportional to h to ensure (14). The $\mathcal{O}$ relations in most of the paper are thus to be understood for $L\uparrow\infty$ and $h\downarrow 0$, respectively. Finally, substituting $\frac{C}{h}$ for L yields the asymptotic error estimate

$$\|f-I_{\Phi,X}(f_L)\|_{\infty,\Omega}=\mathcal{O}(h^{\sigma-n/2})\,\|f\|_w.$$

The net result is that the quasi-interpolation scheme described above gives rise to an approximation error comparable to the approximation error resulting from interpolating f by a "rough" kernel ψ having native space W_σ and acting on a manifold of dimension n. In the next two sections, we will apply these ideas to the n- sphere and n-torus.

3. ERROR ESTIMATES FOR INTERPOLATION

In this section, we quantify the error estimates for interpolation on the n-torus and n-sphere $\mathbb{S}^n := \{v\in R^{n+1} : \|v\|_2=1\}$.

3.1. *The n-Sphere*

We deal with this case first since the results needed essentially have already been obtained in [2] with an improvement from [3]. The kernels considered here have the form

$$\Phi(p,q)=\sum_{\ell=0}^{\infty}\sum_{k=1}^{N(n,\ell)}\hat{\Phi}(\ell,k)\,Y_{\ell,k}(p)\,\overline{Y_{\ell,k}(q)},\qquad p,q\in\mathbb{S}^n,\qquad \hat{\Phi}(\ell,k)>0,\tag{15}$$

where the $Y_{\ell,k}$'s are spherical harmonics of order ℓ, and

$$N(n,\ell)=\frac{2\ell+n-1}{\ell}\binom{\ell+n-2}{\ell-1}=\mathcal{O}(\ell^{n-1}) \qquad \text{for} \quad \ell\geqslant 1.$$

The spherical harmonic $Y_{\ell,k}$ is an eigenfunction of the Laplace–Beltrami operator on $\mathbb{S}^n$ corresponding to the eigenvalue $\lambda_\ell=\ell(\ell+n-1)$, $\ell\geqslant 0$. The set $\{Y_{\ell,k}\}_{k=1}^{N(\ell,n)}$ is chosen to be an orthonormal basis for $\mathscr{E}_\ell$, the eigenspace of the Laplace–Beltrami operator on $\mathbb{S}^n$ corresponding to the eigenvalue λ_ℓ. Collectively, the $Y_{\ell,k}$'s form an orthonormal basis for $L_2(\mathbb{S}^n)$. For such kernels, we have the following useful distance estimates, which were established in [2] with an improvement in [3].

PROPOSITION 3.1. *Let X be any point set on $\mathbb{S}^n$ with mesh norm h, and let Φ be as in* (15). *If for some $\alpha>1$ we have*

$$\hat{\Phi}(\ell)\,N(n,\ell)\leqslant c_1(1+\ell)^{-\alpha}, \qquad \textit{where} \quad \hat{\Phi}(\ell):=\max_{1\leqslant k\leqslant N(n,\ell)}\hat{\Phi}(\ell,k),$$

then

$$\|f-I_{\Phi,X}(f)\|_{\infty,\Omega}^2\leqslant Ch^{\alpha-1}\,\|f\|_\Phi^2,$$

for all f in the native space for Φ, and where the constant C is independent of X.

Proof. The estimate given above is essentially the one found in [2, Corollary 2], *except* that the "M" there is replaced by a constant (cf. [3, Remark 11]). Also, $(1+\Lambda)$ is replaced by the reciprocal of the mesh norm h. ∎

COROLLARY 3.2. *Let X be any point set on $\mathbb{S}^n$ with mesh norm h. If $\hat{\Phi}(\ell)=\mathcal{O}(\frac{1}{\ell^{2\tau}})$ for some $\tau>\frac{n}{2}$ and $\ell\to\infty$, then*

$$\|f-I_{\Phi,X}(f)\|_{\infty,\Omega}=\mathcal{O}(h^{\tau-n/2})\,\|f\|_\Phi.$$

Proof. Since $N(n,\ell)=\mathcal{O}(\ell^{n-1})$, we have that

$$\hat{\Phi}(\ell)\,N(n,\ell)=\mathcal{O}(\ell^{n-1-2\tau}), \qquad \ell\to\infty.$$

Apply Proposition 3.1, with $\alpha=2\tau+1-n$, and then take squares roots of both sides to obtain the desired estimate. ∎

3.2. *The n-Torus*

The case of the n-torus $\mathbb{T}^n$ has been touched on in [1], but not in the generality we require. To handle it, we will use an approach that is similar to the one used in [2] to establish results for the sphere.

The eigenfunctions for $\mathbb{T}^n$ are $\varphi_k(x) = (2\pi)^{-n/2} \exp(ik \cdot \theta_x)$, where $k \in \mathbb{Z}^n$ is a multi-index and θ_x is a vector of n angular coordinates for x. The admissible kernels $\Phi(x, y)$ in (3) are the ones that satisfy $\sum_{k \in \mathbb{Z}^n} \hat{\Phi}(k) < \infty$.

We begin with the following result, which is essentially from [1].

PROPOSITION 3.3. *Let $\Phi(x, y)$ be an admissible kernel on $\mathbb{T}^n$, $\|f\|_\Phi < \infty$, M be a positive integer, and let $X = \{x_1, ..., x_N\}$ be a given knot set. If for each fixed $x \in \mathbb{T}^n$ there are coefficients $c_1, ..., c_N$ such that*

$$\varphi_k(x) = \sum_{j=1}^{N} c_j \varphi_k(x_j), \qquad \|k\|_\infty \leqslant M, \tag{16}$$

and if there is a sequence $b_k < 0$, $\|k\|_\infty = M+1, ...,$ for which

$$\left| \varphi_k(x) - \sum_{j=1}^{N} c_j \varphi_k(x_j) \right|^2 \leqslant b_k$$

holds uniformly in x, then

$$|f(x) - I_{\Phi, X}(f)(x)| \leqslant \|f\|_\Phi \left(\sum_{\|k\|_\infty > M} \hat{\Phi}(k)\, b_k \right)^{1/2}. \tag{17}$$

Proof. The result is a special case of Propositions 3.6 and Theorem 3.8 in [1]. In both results, the set of distributions $\{u_j\}$ is taken to be the set of point evaluations $\{\delta_{x_j}\}$. We remark that the estimate in (17) actually comes from an intermediate step in the proof of Theorem 3.8. ∎

In the case of $\mathbb{T}^n$, the eigenfunctions obviously do not decay at all. Consequently, we cannot expect to find b_k's that decay, and so the best bound we can hope for in (17) will come about if we can bound the b_k's uniformly in $k \in \mathbb{Z}^n$. For obtaining such a bound, it suffices to show that we can find c_j's satisfying (16) and having $\|c\|_{\ell_1} = \sum_j |c_j|$ bounded uniformly in x, N, and M. In [2], the notion of norming set was used to solve an analogous problem. We will need it here as well.

DEFINITION 3.4. Let V be a normed linear space with dual V^*. Given two subspaces $W \subset V$ and $Z \subset V^*$, the set Z is called a *norming set* of W if there exists some $c > 0$ so that

$$\sup_{z \in Z,\, \|z\| = 1} |z(w)| \geqslant C\, \|w\| \qquad \textit{for all} \quad w \in W.$$

Let us specialize this to our situation. We take $V \subset C(\mathbb{T}^n)$ to be the set of all multivariate trigonometric polynomials P of the form

$$P(x) = \sum_{k \in \mathbb{Z}^n,\ \|k\|_\infty \leqslant M} a_k \exp(ik \cdot \theta_x),$$

where θ_x is an n-tuple of angles corresponding to $x \in \mathbb{T}^n$. The linear functionals in Z are point evaluations at the knots. That is, $Z = \{\delta_{x_j}\}_{j=1,\ldots,N}$.

Lemma 3.5. *If the knot set X has mesh norm $h \leqslant 1/2\sqrt{n}\, M$, then Z is a norming set and*

$$\|P|_X\|_\infty \geqslant \tfrac{1}{2}\, \|P\|_\infty.$$

Proof. We will work in periodic coordinates, regarding x and θ_x as being the same. Use the multivariate mean value theorem for scalar-valued functions to write the difference $P(x) - P(y)$ as

$$P(x) - P(y) = \nabla P(\tilde{x}) \cdot (x - y),$$

where $\tilde{x}$ is a point on the shorter of the lines joining x and y. We next estimate the norm of ∇P via Bernstein's univariate inequality applied to each of the n variables separately:

$$\begin{aligned} \sup_{x \in \mathbb{T}^n} \|\nabla P(x)\|_{\ell_2} &\leqslant \left(\sum_{\ell=1}^{n} \|\partial_\ell P\|_\infty^2 \right)^{1/2} \\ &\leqslant \left(\sum_{\ell=1}^{n} M^2\, \|P\|_\infty^2 \right)^{1/2} \\ &\leqslant \sqrt{n}\, M\, \|P\|_\infty. \end{aligned}$$

If $\|x - y\|_2 \leqslant h$, then by this inequality and the previous formula we obtain

$$|P(x) - P(y)| \leqslant \sqrt{n}\, Mh\, \|P\|_\infty$$

Suppose that y is the point at which P attains its maximum; that is, $\|P\|_\infty = |P(y)|$. The mesh norm for X is h; there is thus an $x_j \in X$ for which $\|x_j - y\|_2 \leqslant h \leqslant 1/(2\sqrt{n}\, M)$. Consequently,

$$\|P\|_\infty \leqslant |P(x_j)| + \tfrac{1}{2}\, \|P\|_\infty \leqslant \|P|_X\|_\infty + \tfrac{1}{2}\, \|P\|_\infty.$$

Bringing $\frac{1}{2}\, \|P\|_\infty$ over to left-hand side of this inequality yields the result. ▮

Proposition 3.3 and Lemma 3.5 provide the following estimate.

THEOREM 3.1. *If the knot set X has mesh norm $h \leqslant 1/(2\sqrt{n}\,M)$, then*

$$\|f - I_{\Phi,X}(f)\|_\infty \leqslant 3(2\pi)^{-n/2}\,\|f\|_\Phi \left(\sum_{\|k\|_\infty > M} \hat{\Phi}(k)\right)^{1/2}. \tag{18}$$

Proof. Lemma 3.5 allows us to apply the argument in [2] verbatim to show that there exist c_j's that satisfy (16) and $\sum_{j=1}^N |c_j| \leqslant 2$. From this, it follows that when $\|k\|_\infty > M$,

$$\left|\varphi_k(x) - \sum_{j=1}^N c_j \varphi_k(x_j)\right|^2 \leqslant (2\pi)^{-n}\,(1+2)^2 = 9(2\pi)^{-n} =:\ b_k.$$

The estimate (18) then follows from Proposition 3.3. ∎

We conclude with a result for $\mathbb{T}^n$ analogous to Corollary 3.2 for the n-sphere.

COROLLARY 3.6. *Let X be any point set on $\mathbb{T}^n$ with mesh norm h, and let $\tau > T > n/2$. If $\hat{\Phi}(k) = \mathcal{O}(\|k\|_2^{-2\tau})$, then $\|f - I_{\Phi,X}(f)\|_\infty = \mathcal{O}(h^{\tau - n/2})\,\|f\|_\Phi$.*

Proof. If $M = \lceil (2\sqrt{n}\,h)^{-1} \rceil$ is sufficiently large, then there is a positive constant C such that $\sum_{\|k\|_\infty > M} \hat{\Phi}(k) \leqslant C \sum_{\|k\|_\infty > M} \|k\|_2^{-2\tau}$. If we set

$$N(n, \ell) := \operatorname{card}\{k \in \mathbb{Z}^n : \ell \leqslant \|k\|_2 \leqslant \ell + 1\},$$

then we have

$$\sum_{\|k\|_\infty > M}^{\infty} \hat{\Phi}(k) \leqslant C \sum_{\ell = M+1}^{\infty} \ell^{-2\tau} N(n, \ell).$$

It is easy to show that $N(n, \ell) = \mathcal{O}(\ell^{n-1})$. Using this and $h \sim M^{-1}$ in the previous inequality we arrive at

$$\sum_{\|k\|_\infty > M}^{\infty} \hat{\Phi}(k) \leqslant C \sum_{\ell = M+1}^{\infty} \ell^{n-1-2\tau} = \mathcal{O}(M^{n-2\tau}) = \mathcal{O}(h^{2\tau - n}).$$

Taking square roots then yields $(\sum_{\|k\|_\infty > M}^{\infty} \hat{\Phi}(k))^{1/2} = \mathcal{O}(h^{\tau - n/2})$. The result then follows on inserting this in the estimate in Theorem 3.1. ∎

We remark that if X is quasi-uniformly distributed with mesh norm h, then

$$N = \operatorname{card}(X) = \mathcal{O}(1/h)^n.$$

Thus, in such cases one may rephrase the results above in terms of N.

4. APPROXIMATION ON THE SPHERE AND TORUS

We deal first with the case of the n-sphere. Here, the orthonormal system is based on spherical harmonics, and will be denoted by $\{Y_{\ell,m}\}$ [4, 8]. A function f in $L_2(\mathbb{S}^n)$ has the expansion

$$f = \sum_{\ell=0}^{\infty} \underbrace{\sum_{m=1}^{N(n,\ell)} \hat{f}(\ell, m)\, Y_{\ell,m}}_{\mathbf{P}_\ell f}$$

The truncated version of f is $f_L = \sum_{\ell=0}^{L} \mathbf{P}_\ell f$. We want to estimate $\|f - f_L\|_\infty$.

We will start by estimating the $L_\infty(\mathbb{S}^n)$ norm of the projection $\mathbf{P}_\ell f$. From the addition theorem for spherical harmonics [4, 8], we have

$$\sum_{m=1}^{N(n,\ell)} |Y_{\ell,m}(x)|^2 = \frac{N(n,\ell)}{w_n} P_\ell(n+1; 1) = \frac{N(n,\ell)}{\omega_n},$$

where $P_\ell(n+1; \cdot)$ is the Legendre polynomial of degree ℓ in $n+1$ dimensions, normalized by $P_\ell(n+1; 1) = 1$ (cf. [4]) and ω_n denotes the surface area of $\mathbb{S}^n$. Using this, we get the following bound,

$$\begin{aligned}
\|\mathbf{P}_\ell f\|_\infty &= \max_{x \in \mathbb{S}^n} \left| \sum_{m=1}^{N(n,\ell)} \hat{f}(\ell, m)\, Y_{\ell,m}(x) \right| \\
&\leqslant \left(\sum_{m=1}^{N(n,\ell)} |\hat{f}(\ell, m)|^2 \right)^{1/2} \max_{x \in \mathbb{S}^n} \left(\sum_{m=1}^{N(n,\ell)} |Y_{\ell,m}(x)|^2 \right)^{1/2} \\
&\leqslant \|\mathbf{P}_\ell f\|_2 \sqrt{\frac{N(n,\ell)}{\omega_n}},
\end{aligned}$$

from which it easily follows that

$$\|f - f_L\|_\infty \leqslant \sum_{\ell=L+1}^{\infty} \|\mathbf{P}_\ell f\|_2 \sqrt{\frac{N(n,\ell)}{\omega_n}}. \tag{19}$$

If f belongs to Sobolev space W_σ for the n-sphere, then

$$\|f\|_{W_\sigma}^2 := \sum_{\ell=0}^{\infty} (1 + \lambda_\ell)^\sigma \|P_\ell f\|_2^2 < \infty, \tag{20}$$

where $\lambda_\ell = \ell(\ell + n - 1)$ is an eigenvalue of the Laplace–Beltrami operator for $\mathbb{S}^n$. Consequently, for $f \in W_\sigma$, we can use the Cauchy–Schwarz inequality and (19) to get

$$\|f - f_L\|_\infty \leqslant \left(\sum_{\ell=L+1}^{\infty} (1+\lambda_\ell)^\sigma \, \|\mathbf{P}_\ell f\|_2^2 \right)^{1/2} \left(\sum_{\ell=L+1}^{\infty} \frac{N(n, \ell)}{\omega_n (1+\lambda_\ell)^\sigma} \right)^{1/2}$$
$$\leqslant C \, \|f\|_{W_\sigma} \left(\sum_{\ell=L+1}^{\infty} \ell^{n-1-2\sigma} \right)^{1/2}$$
$$\leqslant L^{\frac{n}{2}-\sigma} C \, \|f\|_{W_\sigma},$$

so that

$$\|f - f_L\|_\infty = \mathcal{O}(L^{\frac{n}{2}-\sigma}) \, \|f\|_{W_\sigma}. \tag{21}$$

Thus α in (7) satisfies $\alpha = \mathcal{O}(L^{\frac{n}{2}-\sigma})$.

We now turn to bounding the Φ-norm in terms of the Sobolev norm. The kernel $\Phi(x, y)$ has the expansion

$$\Phi(x, y) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n, \ell)} \hat{\Phi}(\ell, m) \, Y_{\ell, m}(x) \, \overline{Y_{\ell, m}(y)},$$

where $\hat{\Phi}(\ell, m) > 0$. The decay conditions (4) are now just

$$c(1+\lambda_\ell)^{-\tau} \leqslant \hat{\Phi}(\ell, m) \leqslant C(1+\lambda_\ell)^{-\tau}, \tag{22}$$

and the "inverse" bound in this case is derived via

$$\|f_L\|_\Phi^2 := \sum_{\ell=0}^{L} \sum_{m=1}^{N(n, \ell)} \frac{|\hat{f}(\ell, m)|^2}{\hat{\Phi}(\ell, m)}$$
$$\leqslant c^{-1} \sum_{\ell=0}^{L} (1+\lambda_\ell)^{\tau-\sigma} (1+\lambda_\ell)^\sigma \, \|\mathbf{P}_\ell f\|_2^2$$
$$\leqslant \sup_{0 \leqslant \ell \leqslant L} ((1+\lambda_\ell))^{\tau-\sigma} \, \|f_L\|_{W_\sigma}^2$$
$$= \mathcal{O}(L^{2(\tau-\sigma)}) \, \|f_L\|_{W_\sigma}^2. \tag{23}$$

The quantity γ defined in (11) is thus seen to satisfy $\gamma = \mathcal{O}(L^{\tau-\sigma})$. Recalling Corollary 3.2, we have (under the conditions assumed there) that

$$\|f_L - I_{\Phi, X}(f_L)\|_{\infty, \Omega} = \mathcal{O}(h^{\tau-n/2}) \, \|f_L\|_\Phi := \beta(h, \Phi) \, \|f_L\|_\Phi. \tag{24}$$

From (14), we wish to relate L with h so that

$$\alpha(L, f, \sigma) \approx \gamma(L, \sigma, \Phi) \, \beta(h, \Phi).$$

Inserting the appropriate quantities from (21), (23), and (24) we get the requirement

$$L^{\frac{n}{2}-\sigma} \approx L^{\tau-\sigma} h^{\tau-n/2}.$$

Clearly choosing $L=\mathcal{O}(h^{-1})$ establishes the equivalence with the convergence rate being $h^{\sigma-n/2}$. We summarize the n-sphere result as follows.

THEOREM 4.1. *Let $f\in W_\sigma(\mathbb{S}^n)$, $\Phi\in W_\tau(\mathbb{S}^n)$, and let the finite point set $X\subset\mathbb{S}^n$ have mesh norm h. If $\hat{\Phi}$ satisfies* (22) *with $\tau>\sigma$, then*

$$\operatorname{dist}_{\infty,\mathbb{S}^n}(f, \operatorname*{Span}_{x\in X}\{\Phi(\cdot,x)\})\leqslant C(h^{\sigma-n/2})\,\|f\|_\sigma,$$

where C is independent of $\operatorname{card}(X)$. *Moreover, an approximant to f is given by $I_{\Phi,X}(f_L)$ where $L=\mathcal{O}(h^{-1})$.*

We now turn to approximation on the n-torus. For $f\in\mathbb{T}^n$, the appropriate Sobolev spaces are

$$W_\sigma=\left\{f\in\mathbb{T}^n:\sum_{j\in Z^n}(1+\|j\|_2^2)^\sigma\,|\hat{f}(j)|^2<\infty\right\}. \tag{25}$$

Our cut-off approximant to a given f has the form

$$f_L(x):=\sum_{\|j\|_\infty\leqslant L}\hat{f}(j)\,e^{j\cdot x}.$$

We can thus estimate the approximation error as

$$\begin{aligned}
\|f-f_L\|_{\infty,\Omega}&\leqslant\sum_{\|j\|_\infty>L}\frac{1}{(1+\|j\|_2^2)^{\sigma/2}}(1+\|j\|_2^2)^{\sigma/2}\,|\hat{f}(j)|\\
&\leqslant\left(\sum_{\|j\|_\infty>L}\frac{1}{(1+\|j\|_2^2)^\sigma}\right)^{1/2}\left(\sum_{j\in Z^n}(1+\|j\|_2^2)^\sigma\,|\hat{f}(j)|^2\right)^{1/2}\\
&\leqslant\left(\sum_{\rho=L}^{\infty}\;\sum_{\{j\in Z^n:\,\rho<\|j\|_\infty\leqslant\rho+1\}}\frac{1}{(1+\|j\|_2^2)^\sigma}\right)^{1/2}\|f\|_{W_{2,\sigma}}\\
&\leqslant\left(\sum_{\rho=L}^{\infty}\frac{c\rho^{n-1}}{(1+\rho^2)^\sigma}\right)^{1/2}\|f\|_{W_{2,\sigma}}\leqslant CL^{\frac{n}{2}-\sigma}\,\|f\|_{W_{2,\sigma}}.
\end{aligned}$$

From this we see that the bound in (7) holds, with α satisfying

$$\alpha=\mathcal{O}(L^{\frac{n}{2}-\sigma}). \tag{26}$$

We now turn to computing the quantity γ given in the inverse bound from Eq. (11). Since $\mathscr{I}_L=\{j\in\mathbb{Z}^n:\|j\|_\infty\leqslant L\}$, we have

$$\gamma:=\left(\sup_{\|j\|_\infty\leqslant L}\frac{1}{\hat{\Phi}(j)(1+\|j\|_2^2)^\sigma}\right)^{1/2}.$$

If we assume that the $\hat{\Phi}$'s satisfy the bounds

$$\frac{c_1}{(1+\|j\|_2^2)^\tau} \leqslant \hat{\Phi}(j) \leqslant \frac{c_2}{(1+\|j\|_2^2)^\tau}, \qquad \text{where} \quad \tau \geqslant \sigma, \tag{27}$$

we then have that

$$\gamma^2 \leqslant C \sup_{\|j\|_\infty \leqslant L} (1+\|j\|_2^2)^{\tau-\sigma} \leqslant C L^{2(\tau-\sigma)},$$

so that

$$\gamma = \mathcal{O}(L^{\tau-\sigma}). \tag{28}$$

We will now derive approximation rates for $\mathbb{T}^n$ similar to those derived for $\mathbb{S}^n$.

THEOREM 4.2. *Let $f \in W_\sigma(\mathbb{T}^n)$, $\Phi \in W_\tau(\mathbb{T}^n)$, and let the finite point set $X \subset \mathbb{T}^n$ have mesh norm h. If $\hat{\Phi}$ satisfies* (27) *with $\tau > \sigma$, then*

$$\operatorname{dist}_{\infty, \mathbb{T}^n}(f, \operatorname{Span}_{x \in X} \{\Phi(\cdot, x)\}) \leqslant C(h^{\sigma-n/2}) \|f\|_\sigma,$$

where C is independent of card(X). *Moreover, an approximant to f is given by $I_{\Phi, X}(f_L)$ where $L = \mathcal{O}(h^{-1})$.*

Proof. Note that the assumptions of Corollary 3.6 hold; consequently, we have that

$$\begin{aligned} \|f_L - I_{\Phi, X}(f_L)\|_\infty &= \mathcal{O}(h^{\tau-n/2}) \|f_L\|_\Phi \\ &:= \beta(h, \Phi) \|f_L\|_\Phi. \end{aligned} \tag{29}$$

Recall that in the argument sketched in Section 2, we obtained approximation rates using by choosing L be an appropriate function of the mesh norm h. In the case at hand, α, β, and γ are given by (26), (29), and (28), respectively. By the argument from Section 2, taking $L = \mathcal{O}(h^{-1})$ then gives convergence rates on the order of $h^{\sigma-n/2}$. ∎

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